Overview of my Thesis

My thesis is about limit theorems, which, loosely speaking, guarantee that a sequence of random objects (typically mathematically complicated) differ less and less from a certain limiting random object (typically mathematically easy to handle). Such limit theorems not only lie at the heart of probability theory and statistics, but are also fundamental tools for mathematical modelling and statistical testing throughout all scientific disciplines.

Mathematically, randomness is modelled by random variables (a stock price at a certain point in time, the percentage of a population who will vote for a certain political party, the number of days until a disease is cured etc.) which are said to follow a certain probability distribution. The distribution describes how likely it is that the random variable takes values in a certain range. If one now takes a sequence F_1, F_2, F_3 ,. . ., or short: (F_n), of random variables, the idea of differing less and less from some other, limiting random variable Z is expressed mathematically by saying that the sequence (F_n) *converges* towards the target Z. A typical and very important distribution for the target Z is the Gaussian (often also called normal) distribution, which is ubiquitious in applications and has been studied extensively. Because of its importance, limit theorems in which the target random variable is Gaussian are also called central limit theorems. A very basic example of such a central limit theorem is the tossing of a coin. Here, the nth element F_n of the sequence of random variable is the total number of heads resulting from n coin tosses. One can prove that, as n becomes large, this number eventually follows a Gaussian distribution. To name more interestring examples, there is also strong empirical evidence for central limit theorems to hold when measuring the blood pressure of adult humans, the income of the majority of a population, exchange rates between two currencies and even for quite exotic situations like the length of chess games or internet users' reading time of online articles (all of these examples properly transformed). Limit theorems also appear – somewhat disguised – in the humanities: Whenever a study is conducted on a representative group of individuals, it is an underlying limit theorem which allows one to transfer the drawn conclusions to the whole population.

For my thesis, I worked on three different research projects which are all related to such limit theorems. During my research I mainly focused on the theoretical aspects, although all my results can be and partially already have been applied to concrete situations, either by myself or other researchers. I will now describe these three projects, their scientific impact and several directly related follow-up projects.

In the first project, I investigated the speed with which a sequence of random variables converges to a Gaussian target. This was done in the framework of non-linear transformations of a Gaussian process. More specifically, given a sequence (F_n) of suitable random variables which is known to converge to a Gaussian random variable Z, I was interested in understanding how large the error is if one replaces the nth element F_n by Z. This is important in applications, as one can then choose n large enough so that the error becomes negligible. In mathematical terms, this amounts to bounding a suitable distance $d(F_n, Z)$ between F_n and Z by a deterministic function $f(n)$, whose values become smaller as n grows. Starting from such a bound in dimension one, which was proven by my advisor Prof. Giovanni Peccati and one of his collaborators, I gave a multidimensional extension in full generality. Subsequently, I was also able to provide conditions on when the rate $f(n)$ is optimal, which means that in this case my methods provide the best rates possible. This then led to a technique for decreasing the approximation error by adding a correction term to the sequence (F_n).

The results of this project have been published in the peer-reviewed Latin American Journal of Probability and Mathematical Statistics. I have furthermore presented my findings at several international scientific conferences and summer schools, among others at the University of Cambridge, the University of Barcelona and the University of California, Irvine, as well as several times at the University of Luxembourg.

My second project is about Fourth Moment Theorems. To a random variable F, one can often associate an infinite numerical sequence $m_1(F)$, $m_2(F)$, $m_3(F)$, . . . of so called moments. In many important examples, this sequence of moments completely characterizes the probability distribution of the random variable, i.e. if one knows the values of all the moments, then one can say with absolute certainty that this random variable follows a certain probability distribution. This is for example the case for the Gaussian distribution. One strategy for proving central limit theorem is thus the *method of moments*: Starting from a sequence (F_n) of random variables, calculate the moments $m_1(F_n)$, $m_2(F_n)$, $m_3(F_n)$, ... If these moments all converge to the corresponding moments of a random variable following the Gaussian distribution, then also the distribution of F_n must converge to the Gaussian distribution. This method has the big disadvantage of being computationally quite intensive or worse, in many situations even infeasible to carry out. In 2005, Prof. Giovanni Peccati and Prof. David Nualart from Kansas University proved that in the framework of non-linear transformations of Gaussian processes, the convergence of just the first four moments is often enough to guarantee that the distribution of F_n tends to the Gaussian distribution. In other words, this means that if the first four moments of F_n converge to the ones of the Gaussian distribution, then this is automatically true for all the other moments. This results became known as the Fourth Moment Theorem and, besides being a substantial simplification of the method of moments, turned out to be a very fruitful line of research. It should be pointed out that in particular, the Fourth Moment Theorem proves a central limit theorem. Together with my colleagues Dr. Ehsan Azmoodeh (now at the University of Helsinki) and Dr. Guillaume Poly (now at the University of Rennes 1), we proved that such Fourth Moment Theorems hold in much greater generality than in the Gaussian framework of Peccati and Nualart. Building upon earlier works by Prof. Michel Ledoux from the University Paul Sabatier in Toulouse, we developed a new theory of *Markov chaos* and proved that in this sustantially more general framework Fourth Moment Theorems not only hold for approximation of a Gaussian random variable, but also for random variables following other distributions. As a special case, we obtain a drastically simplified proof of the original Fourth Moment Theorem by Nualart and Peccati. Our theory provides a very deep understanding of the underlying principles, uncovers connections to other branches of mathematics and opens doors to several new lines of research.

The main results of this project have been published in the Journal of Functional Analysis, which is one of the highest ranking scientific journals in functional analysis, while a multidimensional extension has been accepted for publication in Electronic Communications in Probability. Shortly after the main results were published, I was contacted by Prof. Peter Eichelsbacher from the University of Bochum, who is one of the principal investigators of the recently founded Research Training Group 2131, a joint project of the Universities of Bochum, Dortmund and Duisburg-Essen with more than thirty members. He was very excited about the results, invited me for a one week research stay and also asked me to present our results during the opening workshop of the group, where it has met widespread interest. It quickly turned out that our theory can be applied to many problems this research group is considering, and so far we already have very promising preliminary results on Fourth Moment Theorems for the variance gamma distribution which arises in mathematical finance and physics. As he sees great potential of our theory for solving problems his research training group is considering, Prof. Eichelsbacher also asked me to be one of the two lecturers of a summer school which will take place this fall. Together with my former colleague Dr. Guillaume Poly, we have applied our theory to provide a new proof of complex Gaussian product inequality, which is an important result of probability theory. Furthermore, I was recently invited for a two week research stay, combined with a seminar talk about the theory, by Prof. Solesne Bourguin from Boston university. This stay has been very fruitful as well, as it resulted in two new results which currently are in preparation for publication: On the one hand, the techniques developed in my thesis could directly be applied in the framework of non-commutative probability and have led to a very strong quantitative Fourth Moment Theorem in this framework. On the other hand, together with Prof. Murad Taqqu from Boston University and Prof. Nikolai Leonenko from the University of Sheffield, who were both in the audience of my seminar talk, we could apply the theory to provide new Fourth Moment Theorems where the target random variables is an element of the so-called Pearson class. We are currently in the process of preparing two corresponding scientific publications. Last but not least, there are several developments regarding this project at my current university, the university of Rome Tor Vergata. Immediately after arriving there, I was asked to give a research level minicourse about our theory and could win the interest of a master student, who then included it in his masters thesis. With my supervisor Prof. Domenico Marinucci and Dr. Maurizia Rossi (currently at the University of Luxembourg) we are currently applying our methods in the framework of spin random fields and cosmology. To do so, I first had to extend the theory to cover the new setting (this work has already been submitted in a separate research article). Judging by the tremendous positive feedback of the scientific community and the numerous resulting follow-up projects, this part has had the most impact so far.

The third and final project of my thesis deals with local times of Brownian motion. A Brownian motion describes the random motion of particles suspended in a fluid or gas. It is also used in the Black-Scholes-Model of mathematical finance to model the evolution of a stock price over time. The local time, usually denoted by L_t^x, measures how often such a Browian motion visits a certain point x up to time t. In the Black-Scholes example, assuming that x is the stock price in Euros and t is the time in days, L_{10}^{100} thus measures how often this stock attains a price of 100 € over the course of ten days. Beginning in 2010, researchers started to investigate integrated powers of the increments $L_t^{x+h} - L_t^x$ of local time and tried to prove a central limit theorem for them, which is important in order to understand certain charged polymer models in physics. This turned out to be a very hard problem and when I became aware of it, there were already three different proofs for the power two, another two proofs for the power three and a conjecture for the power four published in five different journal articles by a total of seven leading researchers. In my discussions with two of the involved researchers, Professor David Nualart from Kansas University and Professor Jay Rosen from City University of New York, I learned that going to higher powers using the existing methods of proof would at least be very hard if not unfeasible. Consequently, I searched for a different approach and was finally able to prove a limit theorem for an arbitrary power, completely settling the problem and in particular answering the conjecture for the power four (in the affirmative). The earlier results for the powers two and three are contained as special cases in my proof.

The results of this project have been published in the Annals of Probability, one of the highest ranking scientific journals in probability theory. My proof was met with enthusiasm in the scientific community, especially by Prof. Nualart who was very happy that the problem had finally been solved. When presenting my results during the German Probability and Statistics Days in Bochum, I was approached by several researchers, among them Prof. Mark Podolskij from Aarhus University, Denmark. It turned out that my ideas can be combined with some of his findings to prove central limit theorems for integrated powers of more general increments. A corresponding research article is in preparation.

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